Local vs Average Behavior on Inhomogeneous Structures: Recurrence on the Average and a Further Extension of Mermin-Wagner Theorem on Graphs

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Spontaneous breaking of a continuous symmetry cannot occur on a recursive structure, where a random walker returns to its starting point with probability $F = 1$. However, some examples showed that the inverse is not true. We explain this by further extension of the previous theorem. Indeed, even if $F < 1$ everywhere, its average over all the points can be 1. We prove that even on these *recursive on the average* structures the average spontaneous magnetization of $O(n)$ and Heisenberg models is always 0. This difference between local and average behavior is fundamental in inhomogeneous structures and requires a "doubling" of physical parameters such as spectral dimension and critical exponents.

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It is well known that large scale geometry deeply affects the universal behavior of critical systems. Renormalization group theory allows us to understand how this happens for statistical models on regular lattices and predicts that the Euclidean dimension is the only relevant geometrical parameter in determining universality classes.

The situation is not so clear when we are dealing with geometrical structures without translational invariance, such as disordered lattices, fractals, polymers, amorphous materials, or, in general, graphs. In particular we do not know yet which anomalous dimension (if any) is the equivalent of Euclidean dimension for lattices.

Recently [1,2] it has been shown that the critical behavior of continuous symmetry models on graphs is related to the much simpler problem of random walks (RW). Indeed such models cannot have spontaneous symmetry breaking (SSB) at any finite temperature if simple nearest neighbors RW without traps on the same graph are recursive, i.e., if the probability F_i of ever returning to the starting site is 1 for each point i. Since on regular lattices RW are recursive in 1 and 2 dimensions, this result includes the well known Mermin-Wagner theorem (MWT) [3], according to which no spontaneous breaking of a continuous symmetry is possible on 1 and 2 dimensional lattices. Moreover, since the long time behavior of RW probability of returning to the origin allows us to define an anomalous dimension $\tilde{d}$ called *spectral dimension*, [4,5] equal to the Euclidean one d on lattices and ≤ 2 for recursive graphs, it seemed likely that $\tilde{d}$ could be the right generalization of d to graphs. In addition, it is established that MWT can be inverted, i.e., on lattices continuous symmetries are always spontaneously broken in $d > 2$ [6]. Thus one would expect the same should happen for graphs with $\tilde{d} > 2$, i.e., on *transient* graphs, where $F_i < 1$ for each i.

However, an infinite class of counterexamples has been found containing transient fractals with arbitrary large $\tilde{d}$ and showing no SSB [2,7–10]. This fact shows that the previous generalization of MWT cannot be inverted and that $\tilde{d}$ is not a satisfactory generalization of d.

In this Letter we prove a further generalization of MWT, including the previous one, explaining all known counterexamples and suggesting a different definition of $\tilde{d}$, that can be shown to be equivalent to the fracton dimension, eliminating many troubles arising from the previous one.

The fundamental concept we introduce here, i.e., recurrence on the average, is only one aspect of a much more general phenomenon happening on infinite inhomogeneous systems. In fact, on infinite geometrical structures where not all sites are equivalent to each other, the average over all sites i of a local quantity a_i can show completely different properties with respect to each a_i. This is the case for the probability F_i in the above mentioned counterexamples. In fact, even if $F_i < 1$ for each i, F_i can be arbitrarily close to 1, which is an accumulation point for the F_i. If it is the only accumulation point, it follows that the mean value $\bar{F}$ of F_i is exactly 1 [11]. When $\bar{F} = 1$ we shall call RW and the graph where they are defined *recursive on the average* (ROA). Usual recursive structures are also ROA, so that all properties we shall prove for ROA also hold for them.

In the following we shall prove that on ROA structures $O(n)$ and quantum Heisenberg ferromagnetic models cannot show spontaneous average magnetization at any finite temperature. Since we shall use the same notation and some results of Ref. [1], here we briefly recall the mathematical setting of the problem already presented there.

Our discrete structure is described by an infinite graph G. We need an infinite network in order to allow, in principle, SSB. This graph is the limit of a sequence of finite subgraphs chosen in such a way that every bond of G does not belong only to a finite number of subgraphs in the sequence. This is the usual and correct way used in mathematical literature to recover the thermodynamic limit. A particular choice of subgraphs, consisting of fixing an origin point O and taking all sphere subgraphs with center O (set of points and bonds at a chemical

distance from O less than a fixed one), gives a more familiar picture for physicists, recalling the Van Hove thermodynamic limit. Notice that it can be easily proven that the final result is independent of O as well as of the site ordering in each sphere subgraph.

Therefore we start with a generic subgraph in the sequence, i.e., a finite connected graph $\mathcal{G}$ consisting of N sites $i = 1, \ldots, N$ and of bonds $\{i, j\}$ joining them; we say that two sites i and j connected by a bond are nearest neighbors. The graph topology is described by its adjacency matrix $\mathbf{A}$, whose elements are given by

$$A_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are nearest neighbors}, \\ 0 & \text{if } i \text{ and } j \text{ are not nearest neighbors}. \end{cases} \quad (1)$$

We now consider an $O(n)$ model ($n \geq 2$) on $\mathcal{G}$ defined by the Hamiltonian

$$H = -\sum_{ij} \sum_{\mu=1}^{n} \sigma_i^\mu J_{ij} \sigma_j^\mu - \sum_i \sum_{\mu=1}^{n} h^\mu \sigma_i^\mu \quad (2)$$

with the spin components satisfying the constraint

$$\sum_{\mu=1}^{n} \sigma_i^\mu \sigma_i^\mu = 1 \quad (3)$$

and with the ferromagnetic couplings

$$J_{ij} = J_{ji} \begin{cases} = 0 & \text{if } A_{ij} = 0, \\ > 0 & \text{if } A_{ij} = 1. \end{cases} \quad (4)$$

If we introduce the further constraints on the positive couplings

$$J_{ij} \geq d > 0, \quad (5)$$

$$d_i \equiv \sum_j J_{ij} \leq J < \infty \quad (6)$$

for each bond $\{i, j\}$ and each site i of $\mathcal{G}$, all results will depend only on graph topology and not on the particular distribution of $\{J_{ij}\}$. If we allow 0 or ∞ to be accumulation points for the $\{J_{ij}\}$ distribution, the specific form of the latter will affect the long time behavior of RW and to speak of recursive graphs will make no more sense. The former case has been considered in Ref. [1]; the latter has been included in Ref. [2].

Let us choose the magnetic field $\vec{h}$ direction as the nth coordinate axis in the spin space and define $h \equiv |\vec{h}|$. We then define the magnetization per site M, which is the order parameter of the model, as

$$M(h) \equiv \frac{1}{N} \sum_i \langle \sigma_i^n \rangle \equiv \frac{1}{N} \sum_i m_i, \quad (7)$$

where the average is with respect to the usual Boltzmann weight.

In Ref. [1] we showed that, even after taking the thermodynamic limit $N \to \infty$, i.e., on the infinite graph G, $\lim_{h \to 0} M(h) = 0$ if RW on G are recursive. Here we extend this statement to recursive on the average RW.

Consider, as in [1], discrete time RW on $\mathcal{G}$ defined by the hopping probabilities from i to j,

$$S_{ij}(h) \equiv \frac{J_{ij}}{h + d_i}, \quad (8)$$

and a decay or trapping probability at i given by

$$p_i(h) = 1 - \frac{d_i}{d_i + h}. \quad (9)$$

Call $P_{ii}(t; h)$ the probability of returning to the starting site i and $F_{ii}(t; h)$ the probability of returning to i for the first time after t steps, with hopping probabilities (8). Introducing the generating functions $\tilde{f}(\lambda) \equiv \sum_{t=0}^{\infty} \lambda^t f(t)$, we have [12]

$$\tilde{P}_{ii}(\lambda; h) = \frac{1}{1 - \tilde{F}_{ii}(\lambda; h)}. \quad (10)$$

Now $F_i \equiv \tilde{F}_{ii}(\lambda = 1)$ represents the total probability of returning to i at any time. If $F_i = 1$ for a site i, then $F_j = 1$ for every site j. In this case RW are called recursive.

In Ref. [1] we obtained the inequality

$$\beta^{-1}(n - 1) \frac{d}{J^2} \frac{\langle m_i \tilde{P}_{ii}(1; h) \rangle_{\mathcal{G}}^2}{\langle \tilde{P}_{ii}(1; h) \rangle_{\mathcal{G}}} \leq 1, \quad (11)$$

holding for every subgraph $\mathcal{G}$ and so also for the infinite graph G, with $\langle f \rangle_G \equiv \lim_{N \to \infty} N^{-1} \sum_{i=1}^{N} f_i$. Here and in the following, $\lim_{N \to \infty}$ means we are taking the limit according to the above mentioned Van Hove sequence. In this way we avoid any ambiguity about its uniqueness. In general, for simplicity sake, we are assuming that all the limits exist. From a more formal point of view, one should work with "limsup" instead of "lim" in the enumerator of (11); the final result concerning the absence of spontaneous average magnetization holds in both cases.

When RW without traps are recursive, from (10) $\tilde{P}_{ii}(\lambda = 1; h \to 0) \to \infty$ and this fact and (11) allow us to extend the MWT to recursive graphs [1].

Now let us consider recursive on the average graphs, i.e., such that

$$\overline{F} = \langle \tilde{F}_{ii}(\lambda = 1, h = 0) \rangle_G = 1. \quad (12)$$

This class includes all recursive structures as well as many transient graphs. Even in this case it is possible to show that (11) implies the absence of spontaneous magnetization. Here we give the fundamental points of the proof. The complete technical aspect will be discussed in details in a forthcoming review paper [13].

First, we need a definition and a lemma. We define the *measure* $|G'|$ of a subgraph G' of G as the average over G of its characteristic function, i.e., $|G'| = \langle \chi_{G'}(i) \rangle_G$, with $\chi_{G'}(i) = 1$ if $i \in G'$ and 0 else. Then it is possible to prove the following lemma: If $G = \bigcup_{k=1}^{n} G_k$ and the G_k have no common sites, then $\langle A_i \rangle_G = \sum_{k=1}^{n} |G_k| \langle A_i \rangle_{G_k}$, where A_i is a generic function defined on the sites of G.

Now two cases are possible: or for some $\epsilon > 0$ there exists a subgraph G'_ϵ with $|G'_\epsilon| > 0$ such that for $h \to 0$ $m_i > \epsilon$ for $i \in G'_\epsilon$, or such subgraph does not exist for

any positive ϵ. In the latter case one can easily show that the average magnetization $M(h)$ goes to 0 for $h \to 0$, since it must be less than any finite ϵ. In the following we shall prove that the former case is impossible. Let us suppose G'_ϵ exists. From (12) it follows that, for each G' such that $|G'| > 0$, $\lim_{h\to 0}\langle \tilde{P}_{ii}(1;h)\rangle_{G'} = \infty$. If $|G'_\epsilon| = 1$, the averages on G coincide with the averages on G'_ϵ and since $\langle m_i \tilde{P}_{ii}(1;h)\rangle_{G'_\epsilon} \geq \epsilon |G'_\epsilon| \langle \tilde{P}_{ii}(1;h)\rangle_{G'_\epsilon}$, (11) is not satisfied due to the divergence of its right hand side for $h \to 0$.

Consider now $0 < |G'_\epsilon| < 1$. Defining $\overline{G}'_\epsilon \equiv G - G'_\epsilon$ and applying the previous lemma to (11) we get

$$\frac{(\langle m_i \tilde{P}_{ii}(1;h)\rangle_{G'_\epsilon}|G'_\epsilon| + \langle m_i \tilde{P}_{ii}(1;h)\rangle_{\overline{G}'_\epsilon}|\overline{G}'_\epsilon|)^2}{\langle \tilde{P}_{ii}(1;h)\rangle_{G'_\epsilon}|G'_\epsilon| + \langle \tilde{P}_{ii}(1;h)\rangle_{\overline{G}'_\epsilon}|\overline{G}'_\epsilon|} \leq \beta \frac{J^2}{d(n-1)}, \tag{13}$$

which implies

$$\frac{\epsilon^2 \langle \tilde{P}_{ii}(1;h)\rangle_{G'_\epsilon}|G'_\epsilon|}{1 + \frac{\langle \tilde{P}_{ii}(1;h)\rangle_{\overline{G}'_\epsilon}|\overline{G}'_\epsilon|}{\langle \tilde{P}_{ii}(1;h)\rangle_{G'_\epsilon}|G'_\epsilon|}} \leq \beta \frac{J^2}{d(n-1)}. \tag{14}$$

Now, since the enumerator diverges for $h \to 0$, to satisfy (14) at finite temperature even the denominator has to diverge, so we have

$$\lim_{h\to 0} \frac{\langle \tilde{P}_{ii}(1;h)\rangle_{\bar{G}'_\epsilon}}{\langle \tilde{P}_{ii}(1;h)\rangle_{G'_\epsilon}} = \infty. \tag{15}$$

Now notice that (11) can be easily proven not only for constant h, but also for an arbitrary distribution of positive h_i, i.e., for a site dependent magnetic field, by the same steps used in [1] to prove (11). So, we can rescale h on $\bar{G}'_\epsilon$, introducing a field $h'_i(h)$ in such a way that the limit (15) is 1 instead of ∞ with the numerator and the denominator still diverging. Since this rescaling requires larger magnetic fields, by GKS inequalities [14] it follows that all local and average magnetizations of the modified system cannot be smaller than the corresponding quantities in the original one. But if the limit (15) is 1, (11) implies $\epsilon = 0$ at any finite temperature. So the original hypothesis is false and consequently $\lim_{h\to 0} M(h) = 0$.

The same result is easily proven in the same way also for the quantum Heisenberg models considered in Ref. [1], since even for these models an inequality equivalent to (11) has been obtained [1].

So the previous generalization of MWT to recursive graphs has now been extended to a larger class of structures, which can also be transient, provided they are recursive on the average. Notice that although the local recurrence conditions $F_i = 1$ and $\tilde{P}_{ii}(\lambda = 1, h \to 0) = \infty$ are equivalent by (10), this is not the case for the average ones. In fact, even if $\overline{F} = 1$ implies $\langle \tilde{P}_{ii}(1, h \to 0)\rangle_G = \infty$, the inverse is not necessarily true. Our definition of ROA is given by (12), and it is necessary to prove the theorem, since only this definition implies the divergence on the average of $\tilde{P}_{ii}(1, h \to 0)$ on every subgraph with nonzero measure. On the other hand, all the known examples of structures where $\langle \tilde{P}_{ii}(\lambda = 1, h \to 0)\rangle_G = \infty$ but (12) does not hold, show SSB at finite temperature [13].

Thus, although a mathematical proof does not yet exist, we conjecture that continuous symmetry models always have a broken symmetry phase at finite temperature on *transient on the average* graphs, i.e., on graphs where (12) is not true.

Notice that our definition of SSB is based on the average magnetization M. However, on inhomogeneous structures (IS) it is possible to have $M = 0$ but all local magnetization $m_i > 0$ (if 0 is the only accumulation point for the m_i) [13]. This is another consequence of the splitting between local and average behavior on IS. In general, critical phenomena on IS should be classified according to both local and average behavior, and two different classes of critical exponents should be given to describe them completely. Even from an experimental point of view, one should consider the possibility of having different results studying localized or bulk physical quantities. This is particularly important in disordered systems and polymer physics, where IS often occur. This phenomenon also explains why the spectral dimension $\tilde{d}$ defined according to the large time behavior of $P_{ii}(t, h = 0)$ is not a complete generalization of d in the study of SSB. In fact $\tilde{d}$ is defined by

$$P_{ii}(t) \sim t^{-\tilde{d}/2}, \tag{16}$$

and it has been shown that such a definition is independent of the starting point i [5]. However, $\langle P_{ii}(t)\rangle_G$ can have a different asymptotic behavior, so we have to define also an *average spectral dimension* $\tilde{d}_a$ by

$$\langle P_{ii}(t)\rangle_G \sim t^{-\tilde{d}_a/2}. \tag{17}$$

Such a dimension is not greater than the local one and on all the above mentioned counterexamples of transient fractals without SSB it is exactly 1 [11]. Moreover, it can be shown [11] that $\tilde{d}_a$ coincides with the fracton dimension defined according to the low frequency density of vibrational modes [4], i.e.,

$$\rho(\omega) \sim \omega^{\tilde{d}_a - 1} \tag{18}$$

for $\omega \to 0$.

In general many other physical parameters, such as the intrinsic fractal dimension, on IS split in a local and an average one, coinciding only in particular cases. A further analysis of these phenomena will be presented in forthcoming papers.

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