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Diffusion on nonexactly decimable tree-like fractals

Raffaella Burioni†‡‖, Davide Cassi†§¶, Alberto Pirati§ and Sofia Regina†§+

† Istituto Nazionale di Fisica Della Materia, Unitá di Parma, Italy

‡ Dipartimento di Fisica, Università di Milano, via Celoria 16, 20133 Milano, Italy

§ Dipartimento di Fisica, Università di Parma, Viale delle Scienze, 43100 Parma, Italy

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Abstract. We calculate the spectral dimension of a wide class of tree-like fractals by solving the random walk problem using a new analytical technique, based on invariance under generalized cutting-decimation transformations. These fractals are generalizations of the NT_D lattices and they are characterized by noninteger spectral dimension equal to or greater than 2, nonanomalous diffusion laws, dynamical dimension splitting and the absence of phase transitions for spin models.

1. Introduction

The spectral dimension $\tilde{d}$ of non translation-invariant structures is up to now the best generalization of the Euclidean dimension of lattices when dealing with dynamical and thermodynamical properties. It can be defined according to the large-time asymptotic behaviour of random walks and it can be shown to be relevant for several different physical phenomena, such as vibrational dynamics, electrical conductivity and phase transitions. Many properties of $\tilde{d}$ have been argued and proved starting from exact analytical calculations on a few particular cases and any further progress in understanding its relevance strongly depends on the availability of exact results in a wider range of particular cases. The most used mathematical tools to calculate $\tilde{d}$ belong to two main classes: renormalization group and combinatorial techniques. Unfortunately, renormalization group can give exact results only on exactly decimable fractals which, in turn, have been shown to have $\tilde{d} < 2$. On the other hand combinatorial techniques, based on the iteration of cuttings, can only be applied to discrete structures with a given characteristic scale. Recently, a particular combination of both techniques has been successfully applied to the more complex case of NT_D lattices [1]. These lattices are not exactly decimable but they are invariant under a more complex geometrical transformation we shall call cutting-decimation. On NT_D lattices the random walks problem has been analytically solved using a cutting-decimation method based on a time-rescaling technique [2]. The NT_D has been shown to have remarkable properties such as noninteger spectral dimension equal to or greater than 2, nonanomalous diffusion laws, absence of phase transitions [3] and dynamical dimension splitting [4]. Because of these peculiar characteristics, NT_D lattices have been widely used for the study of statistical mechanics on non translation-invariant lattices and they have opened the way for the research of other structures sharing the same properties.

‖ E-mail address: burioni@pr.infn.it

¶ E-mail address: cassi@pr.infn.it

+ E-mail address: regina@pr.infn.it

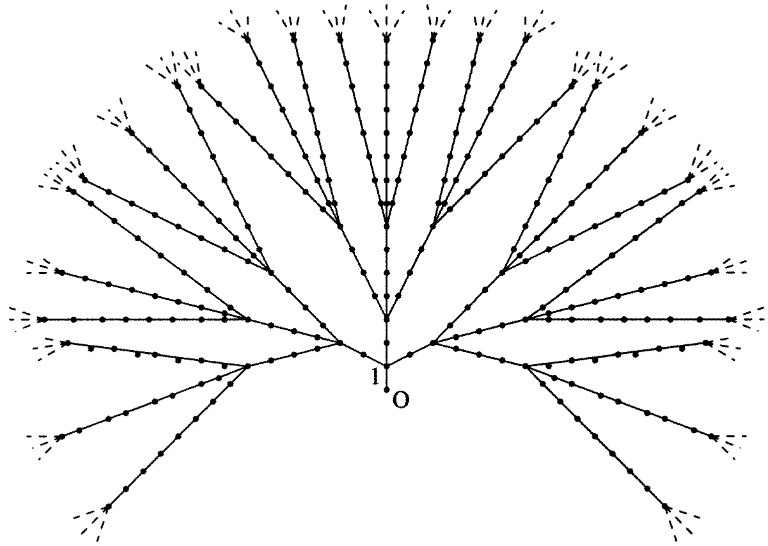

Figure 1. $2^m NT_D$ with $k = 3$, $m = 1$.

In this paper we deal with generalizations of NT_D lattices we shall call $2^m NT_D$, nNT_D and *p-polygon* NT_D introducing three corresponding new techniques to analytically solve the random walks problem. As a result we obtain very general structures with the same properties of NT_D and, at the same time, test new techniques that can be useful in other cases of non exactly decimable fractals.

2. Random walks on $2^m NT_D$ via exact time rescaling

The $2^m NT_D$ are infinite fractal trees that can be recursively built using the following method (figure 1). An origin point O is connected to point A by a link of length 1 (the log of the tree); from A the log splits into k branches of length 2^m (i.e. made of 2^m consecutive links) which, in turn, split into k branches of length 2^{2m} and so on in such a way that each branch of length 2^{nm} splits into k branches of length $2^{(n+1)m}$. The simplest case is that of $m = 1$ and corresponds to the NT_D lattices. The $2^m NT_D$ invariance under the cutting-decimation transform can be described as follows (figure 2). Suppose we cut the log of the tree and separate the k branches starting from A. Then each of these branches is exactly the same as the original tree with a dilatation factor equal to 2^m and therefore it can be reduced to the original tree through decimation. Let us now consider the discrete time random walks problem on $2^m NT_D$, in order to calculate the spectral dimension $\tilde{d}$ of the lattice. The spectral dimension of a graph is defined through the relation [5],

$$P_O(t) \sim t^{-\tilde{d}/2} \quad (1)$$

where $P_O(t)$ is the probability for a random walker to return to the starting point O after a walk of t steps for $t \to \infty$. The cutting transformation applied to random walks gives a relation between $P_O^{\text{tree}}(t)$, the probability of returning to the starting site O after a t-steps walk on the whole tree, and $P_A^{\text{branch}}(t)$, the probability of returning to the starting site A after a t-steps walks on one of the branches starting from A. This relation has been obtained in [2] for the case $m = 1$ in terms of the generating functions (discrete Laplace transforms)

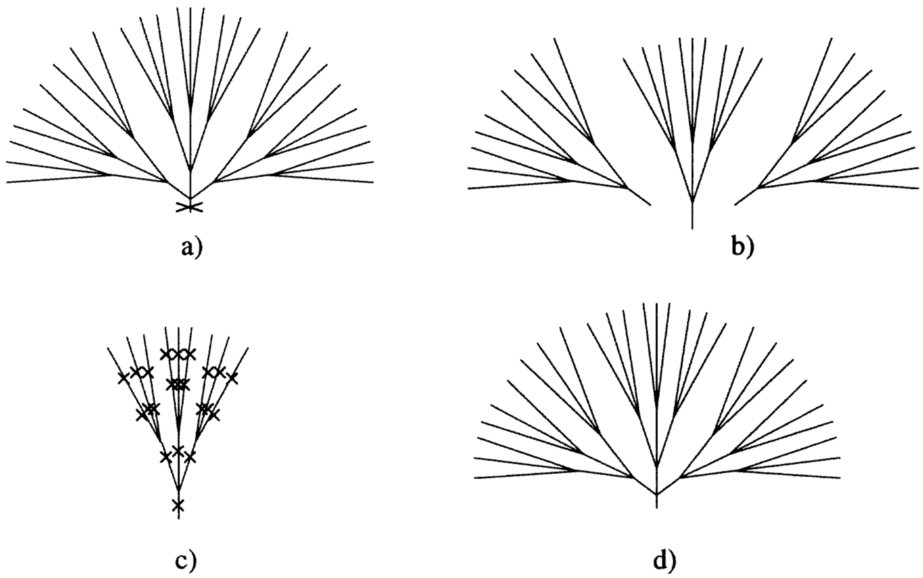

Figure 2. Cutting-decimation procedure: (a) cutting of the log of the NT_D (b) separation of the k branches (c) decimation of the points labelled by X (d) recovering of the original NT_D.

$\widetilde{P}(\lambda)$ of the probabilities $P(t)$, but it holds for $2^m NT_D$ for every m:

$$\widetilde{P_O}^{\text{tree}}(\lambda) = \frac{\widetilde{P_A}^{\text{branch}}(\lambda) + k}{(1-\lambda^2)P_A^{\text{branch}}(\lambda) + k}. \tag{2}$$

In the $m = 1$ case the decimation transformation has been performed using a time-rescaling technique. Indeed, the motion of the random walker on the branch considered only after an even number of steps can be exactly mapped in the motion of a random walker on the tree after the introduction of a staying probability $p_{ii} = \frac{1}{2}$ in every site i. This equivalence can be introduced in the generating functions through the substitutions [2],

$$\widetilde{P_O}(\lambda) \to \frac{\lambda}{2-\lambda}\widetilde{P_O}\left(\frac{2}{2-\lambda}\right) \tag{3}$$

$$\lambda \to \lambda^2. \tag{4}$$

In the case $m = 1$ equations (3) and (4) can be used to rewrite (2) as

$$\widetilde{P_O}^{\text{tree}}(\lambda) = \frac{\frac{2}{2-\lambda^2}\widetilde{P_O}^{\text{tree}}\left(\frac{\lambda^2}{2-\lambda^2}\right) + k}{(1-\lambda^2)\frac{2}{2-\lambda^2}\widetilde{P_O}^{\text{tree}}\left(\frac{\lambda^2}{2-\lambda^2}\right) + k}. \tag{5}$$

If $m > 1$ this procedure must be iterated m times obtaining

$$\widetilde{P_O}^{\text{tree}}(\lambda) = \frac{\left(\prod_{i=1}^{m}\frac{2}{2-\lambda_i^2}\right)\widetilde{P_O}^{\text{tree}}(\lambda_{i+1}) + k}{(1-\lambda^2)\left(\prod_{i=1}^{m}\frac{2}{2-\lambda_i^2}\right)\widetilde{P_O}^{\text{tree}}(\lambda_{i+1}) + k} \tag{6}$$

with

$$\lambda_i = \begin{cases} \lambda & i = 1 \\ \dfrac{\lambda_{i-1}^2}{2-\lambda_{i-1}^2} & i > 1 \end{cases}$$

i being the iteration step. To find the value of $\tilde{d}$ we consider the singularities of the generating functions as $\lambda = 1 - \epsilon$, $\epsilon \to 0^+$. In terms of ϵ the decimation transformation can be resumed as

$$\widetilde{P_A}^{\text{branch}}(\epsilon) \sim 2^m \widetilde{P_O}^{\text{tree}}(2^{2m}\epsilon) \tag{7}$$

as $\epsilon \to 0$, so that (6) becomes

$$\widetilde{P_O}^{\text{tree}}(\epsilon) \sim \frac{2^m \widetilde{P_O}^{\text{tree}}(2^{2m}\epsilon) + k}{2\epsilon 2^m \widetilde{P_O}^{\text{tree}}(2^{2m}\epsilon) + k}. \tag{8}$$

From (8), applying standard Tauberian theorems [2] one obtains for a $2^m NT_D$ graph,

$$\tilde{d}_{2^m} = 1 + \frac{\ln k}{\ln 2^m} \tag{9}$$

which represents the generalization of the result obtained for $m = 1$.

3. Random walks on nNT_D and *p-polygon* NT_D via asymptotic decimation

The previous results can be extended to nNT_D, where n is now an integer and not necessarily a power of 2, and to *p-polygon* NT_D, where the branches of NT_D are replaced by p-vertices regular polygons (figure 3). Let us consider nNT_D first. While relation (2) still holds, the exact time-rescaling procedure cannot be applied to the branch of generic length n. However, even in this case it is possible to obtain an asymptotic recursion relation applying the renormalization group techniques usually implemented on exactly decimable fractals [6]. Although this procedure cannot give an exact equation for $\widetilde{P_O}^{\text{tree}}(\lambda)$ as in the previous case, it can nevertheless be used to obtain the exact value of $\tilde{d}$ via an asymptotic expansion. Indeed, in this case the branch of the nNT_D can be considered as a tree with a dilatation factor n. The log of this tree can be reduced to a unitary length log after the suppression of the $n-2$ sites between the vertices and introducing a new link connecting the edges. The same operation can be iterated for branches of every length suppressing the inner

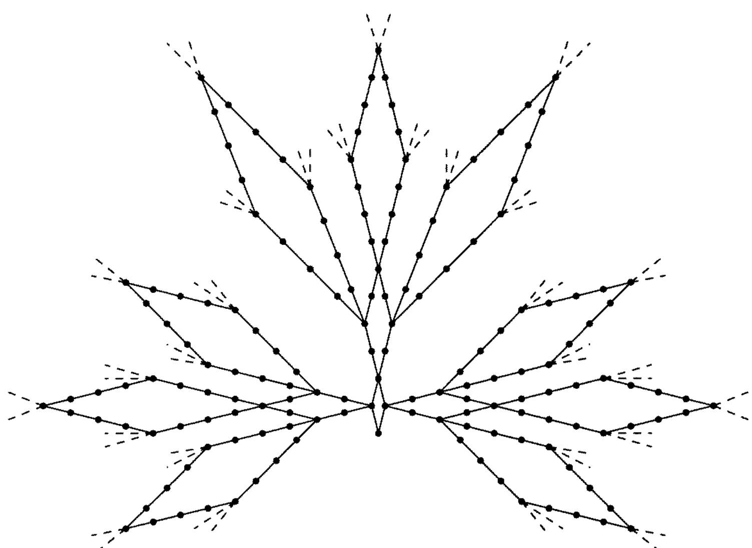

Figure 3. *Four-polygon* NT_D with $k = 1$, $n = 2$.

$n-2$ consecutive sites in every sequence of n sites and introducing a new link between the surviving points. The final structure is equal to the original tree and the generating function $\widetilde{P_A}^{\text{branch}}(\epsilon)$ becomes $\widetilde{P_A}^{\prime\text{branch}}(\epsilon')$ where

$$\epsilon' = n^2\epsilon \tag{10}$$

$$\widetilde{P_A}^{\prime\text{branch}}(\epsilon') = \frac{1}{n}\widetilde{P_A}^{\text{branch}}(\epsilon). \tag{11}$$

Now $\widetilde{P_A}^{\prime\text{branch}}(\epsilon')$ coincides with $\widetilde{P_O}^{\text{tree}}(\epsilon')$ since the original branch is now a tree and (2) can be rewritten as

$$\widetilde{P_O}^{\text{tree}}(\epsilon) = \frac{n\widetilde{P_O}^{\text{tree}}(n^2\epsilon) + k}{2\epsilon n\widetilde{P_O}^{\text{tree}}(n^2\epsilon) + k}. \tag{12}$$

Using the procedure described in the previous section for $2^m NT_D$, from (12) it follows that for an n-NT_D the spectral dimension is given by

$$\tilde{d}_n = 1 + \frac{\ln k}{\ln n}. \tag{13}$$

An analogous technique can be used for *p-polygon* NT_D (figure 3). The log polygon now has p faces of unitary length; from each of $p-1$ of its vertices k polygons depart, whose faces have length n and so on. These structures, though similar to NT_D, are no longer loopless structures nor necessarily bipartite graphs (for example the *three-polygon* tree). The cutting-decimation transform can be applied to *p-polygon* NT_D as in the case of NT_D with the same substitutions (10) and (11). Indeed, even if (2) does not hold in this case, a new relation between the generating functions of the tree and that of one of its branches can be obtained using bundled structures theory [7]. A bundled structure is a composed graph obtained by joining to each point of a 'base graph' a copy of a 'fibre' graph in such a way that every fibre has only one point in common with the base and no points in common with the other fibres. Let us consider a *p-polygon* NT_D and suppose to attach k branches also in the free vertex of the log (the root of the tree): we obtain a bundled structure having the log polygon as a base and the graph made of k branches as the fibre. Since for a *p-polygon*,

$$\widetilde{P_O}(\lambda) \sim \frac{1}{p(1-\lambda)} \tag{14}$$

as $\lambda \to 1$, we obtain for our bundled structure,

$$\widetilde{P_O}^{\text{b.s.}}(\lambda) = \frac{1}{1-\frac{k}{k+1}\widetilde{F_A}^{\text{branch}}(\lambda)}\frac{1}{p}\left(1-\frac{\lambda}{k+1}\frac{1}{1-\frac{k}{k+1}\widetilde{F_A}^{\text{branch}}(\lambda)}\right)^{-1} \tag{15}$$

where $\widetilde{P_O}^{\text{b.s.}}(\lambda)$ is the generating function of the probability of returning to point O (one of the vertices of the log polygon) after a random walk on the bundled structure and $\widetilde{F_A}^{\text{branch}}(\lambda)$ is the generating function of the probability of returning for the first time to the point of connection with the base after a random walk on the fibre. For a generic structure the generating function of the probability of returning to the starting point O, $\widetilde{P_O}(\lambda)$ and of returning for the first time to the starting point, $\widetilde{F_O}(\lambda)$, satisfy [8],

$$\widetilde{P_O}(\lambda) = (1-\widetilde{F_O}(\lambda))^{-1}. \tag{16}$$

Using

$$\widetilde{F_O}^{\text{b.s.}}(\lambda) = \frac{k}{k+1}\widetilde{F_A}^{\text{branch}}(\lambda) + \frac{1}{k+1}\widetilde{F_O}^{\text{tree}}(\lambda) \tag{17}$$

where $F_O{}^{\text{tree}}(\lambda)$ refers to the *p-polygon* NT_D, from equations (15)–(17) a relation between $\widetilde{P_O}{}^{\text{tree}}(\lambda)$ and $\widetilde{P_A}{}^{\text{branch}}(\lambda)$ follows, which represents the cutting transformation. It is now possible to perform the cutting-decimation transform for *p-polygon* NT_D and obtain

$$\tilde{d}_p = 1 + \frac{\ln k(p-1)}{\ln n}. \tag{18}$$

In the same way we can calculate the spectral dimension of an NT_D built with d-dimensional simplexes instead of *p-polygons*. A d-dimensional simplex is a complete graph of $d+1$ points i.e. a graph where each point is the nearest neighbour of all other points. The two-dimensional case is the triangle, the three-dimensional case is the tetrahedron and so on. Since for the d-simplex $\widetilde{P_O}(\lambda) \sim 1/(d+1)(1-\lambda)$ the spectral dimension is

$$\tilde{d}_d = 1 + \frac{\ln kd}{\ln n}. \tag{19}$$

4. Conclusions

The generalized NT_D lattices described here have in general noninteger spectral dimension depending on the geometrical features of the lattices, such as the growing factor 2^m or n, the number of branches k and the number of vertices of the polygon. Since the intrinsic fractal dimension always coincides with the spectral dimension, the diffusion [2] on all of these lattices is not anomalous i.e. is described by the asymptotic law,

$$\langle r^2(t)\rangle \sim t^{\alpha} \tag{20}$$

with $\alpha = 1$. Moreover, as in the case of simple NT_D lattices [4], we can also show that generalized NT_D present dynamical dimension splitting. This means that the vibrational spectral dimension $\bar{d}$, which characterizes the density of vibrational modes as $\omega \to 0$ through the relation,

$$\rho(\omega) \sim \omega^{\bar{d}-1} \tag{21}$$

is different from the diffusive spectral dimension of random walks $\tilde{d}$. In particular we have $\bar{d} = 1$ and this can be intuitively understood noting that the topology of generalized NT_D is dominated by linear chains which become increasingly longer in the outer branches. Since $\bar{d} = 1$, by the generalized Mermin and Wagner theorem [9], phase transitions with spontaneous breaking of a continuous symmetry for nonzero temperature cannot occur on these structures. All of these remarkable properties, typical of simple NT_D lattices, have been obtained here even in the absence of some peculiar characteristics of NT_D such as the bipartite and loopless nature of the graph. This enlarged family of lattices can be used, as NT_D lattices, to reach a better understanding of the geometrical features giving rise to dynamical dimension splitting and, at the same time, they represent the testing ground to study the relation between spectral dimensions and physical phenomena.

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